A Novel Algorithm for Bearing Stiffness Optimization

M. Jokic* University of Zagreb Zagreb, Croatia
M. Stegic† University of Zagreb Zagreb, Croatia
N. Vrankovic‡ University of Zagreb Zagreb, Croatia

Abstract— *This paper presents a novel technique for structural optimization with respect to vibration response. Under the assumptions of asymptotic stability of the state-space system representing the structure, optimization criteria is defined in terms of H_∞ norm of the system. In doing so, special structure of the system matrices is utilized to eliminate some of the optimization variables. To overcome its high computational cost and make it applicable for optimization of large-scale structures, a novel reduced-order optimization algorithm is proposed. A numerical example which clearly illustrates the applicability and efficiency of the proposed optimization procedure is presented.*

Keywords: bearing stiffness optimization, passive vibration control

I. Introduction

The development trends of modern rotating machinery, most notably higher operating speed and increased power/weight ratio, longer operational life, decreased noise, higher demands for reliability, durability as well as safety, have resulted in increased importance of vibration attenuation strategies. Such strategies include passive vibration control, which is commonly achieved by damping, isolation and stiffening. Damping usually involves vibration energy dissipation via fluid dampers, elastomers, hysteretic elements, etc. Prior to dissipation, vibration energy may be transferred to tuned mass dampers, or it may be converted to electrical energy using piezoelectric tranducers and then dissipated or stored (energy harvesting). Isolation and stiffening, however, usually involve some sort of structural elements stiffness optimization in order to prevent vibration propagation or to shift the structure resonant frequency beyond the excitation frequency band.

Recently, there has been an increasing interest in posing the structural vibration optimization problem as an optimal control problem [1], [2], [3]. This approach quantifies the structure vibration response in terms of suitably chosen system norms, and the resulting optimal control problem is subsequently solved using numerical techniques.

The stiffness optimization framework proposed in this paper falls into this broad category of control-oriented methods, i.e. it is based on H_∞ optimality condition for state-space systems. In doing so, it assumes regularity of the structure mass matrix. Potential drawback is its increased computational cost, which is a common issue associated with control-oriented, as well as some other structural optimization methods. To overcome this, a reduced-order optimization technique is proposed as well, thus adapting the original optimization framework to optimization for large-scale structures.

Throughout this paper, we use the the following notation. Let $\mathbf{R}$ denote the set of real numbers and $\mathbf{I}$ is the identity matrix. For a matrix $\mathbf{A}$, we denote by $\mathbf{A}^{\mathrm{T}}$, $\mathbf{A}^*$ and $\sigma_{\max}(\mathbf{A})$ its transpose, conjugate transpose and maximum singular value (spectral norm), respectively. We define $\mathbf{He}\,[\mathbf{A}]$ as an abbreviation for $\mathbf{A}+\mathbf{A}^{\mathrm{T}}$. For a vector $\mathbf{v}$, $\|\mathbf{v}\|$ denotes its Euclidean norm. We use $\mathbf{A} > \mathbf{B}$ ($\mathbf{A} \geq \mathbf{B}$) and $\mathbf{A} < \mathbf{B}$ ($\mathbf{A} \leq \mathbf{B}$) to denote, respectively, positive and negative (semi)definite ordering of symmetric matrices $\mathbf{A}$ and $\mathbf{B}$. A space of all signals $\mathbf{w}(t)$ such that $\int_{t=0}^{\infty} \|\mathbf{w}(t)\|^2 \mathrm{d}t < \infty$ is denoted $L[0,\infty)$.

II. Optimization framework

Consider the following second order linear time invariant system that represents the structure

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{D}\dot{\mathbf{y}}(t) + \mathbf{S}(\mathbf{s})\mathbf{y}(t) &= \mathbf{B}_1\mathbf{w}(t), \\ \mathbf{z}(t) &= \mathbf{C}_1\dot{\mathbf{y}}(t) + \mathbf{C}_2\mathbf{y}(t), \end{aligned} \tag{1}$$

where $\mathbf{M} \in \mathbf{R}^{q\times q}$, $\mathbf{D} \in \mathbf{R}^{q\times q}$ and $\mathbf{S}(\mathbf{s}) \in \mathbf{R}^{q\times q}$ are mass, damping and stiffness matrices, respectively, $\mathbf{B}_1 \in \mathbf{R}^{q\times m}$ is input matrix and $\mathbf{C}_1 \in \mathbf{R}^{p\times q}$ and $\mathbf{C}_2 \in \mathbf{R}^{p\times q}$ are velocity and displacement output matrices, respectively. Time-dependent vectors $\mathbf{y}(t), \dot{\mathbf{y}}(t), \ddot{\mathbf{y}}(t) \in \mathbf{R}^q$, $\mathbf{w}(t) \in \mathbf{R}^m$ and $\mathbf{z}(t) \in \mathbf{R}^p$ are displacement, velocity, acceleration, input and output vectors, respectively. The input vector is force or displacement excitation, and we assume that it is steady-state, periodic and $\mathbf{w}(t) \in L[0,\infty)$. Structure stiffness matrix is assumed to be affine function of the stiffness parameters as follows

$$\mathbf{S}(\mathbf{s}) = \mathbf{S}_0 + \sum_{i=1}^{l} s_i \mathbf{S}_i, \tag{2}$$

where

$$\mathbf{s} = \{s_i \mid i = 1, \ldots, l\}. \tag{3}$$

In order to accommodate actual stiffness parameters constraints due to design, technology and other requirements, we introduce the following general equality and inequality

*mjokic@fsb.hr
†mstegic@fsb.hr
‡nvrankov@fsb.hr

constraints

$$\begin{array}{rcl}\mathbf{h}(\mathbf{s}) & = & \mathbf{0}, \\ \mathbf{g}(\mathbf{s}) & \leq & \mathbf{0}.\end{array} \tag{4}$$

Obviously, such constraints define feasible set for stiffness parameters, as well as for stiffness matrices. Furthermore, we assume that $\mathbf{M} > \mathbf{0}$ and $\mathbf{S}(\mathbf{s}) \geq \mathbf{0}$ for all $\mathbf{s}$ that satisfy the constraints (4).

The system (1) may be rewritten as a state space system

$$\begin{array}{rcl}\dot{\mathbf{x}}(t) & = & \mathbf{A}(\mathbf{s})\mathbf{x}(t) + \mathbf{B}\mathbf{w}(t), \\ \mathbf{z}(t) & = & \mathbf{C}\mathbf{x}(t),\end{array} \tag{5}$$

where

$$\mathbf{A}(\mathbf{s}) = \begin{pmatrix} -\mathbf{M}^{-1}\mathbf{D} & -\mathbf{M}^{-1}\mathbf{S}(\mathbf{s}) \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \tag{6}$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{M}^{-1}\mathbf{B}_1 \\ \mathbf{0} \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{pmatrix}, \tag{7}$$

and the system state vector is

$$\mathbf{x}(t) = \begin{pmatrix} \dot{\mathbf{y}}(t)^{\mathrm{T}} & \mathbf{y}(t)^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}. \tag{8}$$

Assume that the system (1) is asymptotically stable for all $\mathbf{s}$ that satisfy the constraints (4), i.e. all eigenvalues of $\mathbf{A}(\mathbf{s})$ lie within open left half of the complex plane. This implies that $\mathbf{z}(t) \in L[0,\infty)$ for all $\mathbf{w}(t) \in L[0,\infty)$. Furthermore, stiffness parameter optimization problem may be cast as the H_∞ optimal control problem, namely to find $\mathbf{s}$ that minimize a real scalar γ such that input and output signals of the system (1) satisfy

$$\int_{t=0}^{\infty} \|\mathbf{z}(t)\|^2 \mathrm{d}t < \gamma^2 \int_{t=0}^{\infty} \|\mathbf{w}(t)\|^2 \mathrm{d}t \tag{9}$$

for all $\mathbf{w}(t) \in L[0,\infty)$.

Frequency domain equivalent of the inequality (9), reffered to as the bounded real lemma (BRL), is the system H_∞ norm condition

$$\|\mathbf{G}\|_\infty = \sup_{\omega \in \mathbf{R}\cup\infty} \sigma_{\max}\left(\mathbf{G}(\mathrm{i}\omega)\right) < \gamma, \tag{10}$$

where $\mathbf{G}(\mathrm{i}\omega) = \mathbf{C}(\mathrm{i}\omega - \mathbf{A}(\mathbf{s}))^{-1}\mathbf{B}$ is the system frequency response. Equivalently, (10) can be written as the matrix inequality

$$\begin{pmatrix} \mathbf{I} \\ \mathbf{G}(\mathrm{i}\omega) \end{pmatrix}^* \begin{pmatrix} -\gamma^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{G}(\mathrm{i}\omega) \end{pmatrix} < \mathbf{0} \tag{11}$$

for all $\omega \in \mathbf{R} \cup \infty$.

The scalar γ essentially quantifies the worst-case gain of the system, or in other words, largest ratio of Euclidean norms of output and input signal amplitudes for all steady-state sinusoidal input/output signals and all frequencies.

A. Dissipativity inequalities and optimization criteria

Kalman-Yakubovic-Popov lemma is the fundamental result in dynamical systems theory that establishes equivalence between the frequency domain inequality (11) and a linear matrix inequality (LMI) for the system state space realization [4]. Thus, the conditon for the bounded realness of the system (1) may be expressed in terms of LMI involving system matrices (6) and (7), rather than as infinitely many inequalities (11) parametrized by ω, as follows.

The condition (11) holds true if and only if there exists a matrix $\mathbf{X} = \mathbf{X}^{\mathrm{T}} \in \mathbf{R}^{2q\times 2q}$ that satisfies the following LMI

$$\begin{pmatrix} \mathbf{A}(\mathbf{s})^{\mathrm{T}}\mathbf{X} + \mathbf{X}\mathbf{A}(\mathbf{s}) & \mathbf{X}\mathbf{B} & \mathbf{C}^{\mathrm{T}} \\ \mathbf{B}^{\mathrm{T}}\mathbf{X} & -\gamma\mathbf{I} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} & -\gamma\mathbf{I} \end{pmatrix} < \mathbf{0}. \tag{12}$$

Due to the fact that we are dealing with the second order system, which yields the special structure of the matrices (6) and (7), some of the variables in $\mathbf{X}$ may be eliminated. We partition $\mathbf{X}$ according to the block structure of $\mathbf{A}(\mathbf{s})$ into

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{X}_2 \\ \mathbf{X}_2^{\mathrm{T}} & \mathbf{X}_4 \end{pmatrix} \in \mathbf{R}^{2q\times 2q} \tag{13}$$

and apply Lemma 1 (see Appendix) to eliminate the blocks containing $\mathbf{X}_4$. This results in the following LMIs

$$\begin{pmatrix} \mathbf{He}\left[-\mathbf{X}_1\mathbf{M}^{-1}\mathbf{D} + \mathbf{X}_2\right] & \mathbf{X}_1\mathbf{M}^{-1}\mathbf{B}_1 & \mathbf{C}_1^{\mathrm{T}} \\ (\mathbf{M}^{-1}\mathbf{B}_1)^{\mathrm{T}}\mathbf{X}_1 & -\gamma\mathbf{I} & \mathbf{0} \\ \mathbf{C}_1 & \mathbf{0} & -\gamma\mathbf{I} \end{pmatrix} < \mathbf{0}, \tag{14}$$

$$\begin{pmatrix} \mathbf{He}\left[-\mathbf{S}(\mathbf{s})\mathbf{X}_2\right] & \mathbf{X}_2^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{B}_1 & \mathbf{C}_2^{\mathrm{T}} \\ (\mathbf{M}^{-1}\mathbf{B}_1)^{\mathrm{T}}\mathbf{X}_2 & -\gamma\mathbf{I} & \mathbf{0} \\ \mathbf{C}_2 & \mathbf{0} & -\gamma\mathbf{I} \end{pmatrix} < \mathbf{0}. \tag{15}$$

Finally, the H_∞ optimal control problem at hand may be formulated as

$$\min_{\mathbf{s}} \gamma \tag{16}$$

such that

1. stiffness parameters $\mathbf{s}$ satisfy the constraints (4),
2. there exist $\mathbf{X}_1 = \mathbf{X}_1^{\mathrm{T}}, \mathbf{X}_2 \in \mathbf{R}^{q\times q}$ such that (14) and (15) hold true.

B. Numerical optimization procedure

Due to the fact that both $\mathbf{S}(\mathbf{s})$ and $\mathbf{X}_2$ in the top left block of (15) are variables, inequality (15) is bilinear matrix inequalities (BMI), which renders (16) a nonconvex optimization problem. Rather than applying numerically very expensive global optimization procedure for tackling such problem, we opt for a local optimization using readily available BMI optimization software [5]. Obviously, such local optimization procedure yields the result that depends on the initial guess, as well as the constraints (4).

The motivation for such choice, apart from avoiding extensive numerical calculations, is that good initial guess for stiffness parameters is usually available — for example, initial parameters may be tuned to some sub-optimal configuration by means of some readily-available technique, or we may be dealing with some pre-existing sub-optimal design that needs further optimization. Additionally, stiffness parameters may be constrained to some small feasible set due to, for example, design requirements, in which case the local optimization hopefully results in finding global optimum.

III. Reduced-order optimization for large-scale systems

Bounded realness condition, expressed it terms of inequalities (14) and (15), imposes the existence of matrices $\mathbf{X}_1 = \mathbf{X}_1^{\mathrm{T}}, \mathbf{X}_2 \in \mathbf{R}^{q\times q}$, whose dimensions q are equal to the dimensions of the mass, damping and stiffness matrices. Thus, additional $2q^2 + q$ variables are introduced. Furthermore, although $\mathbf{M}, \mathbf{D}$ and $\mathbf{S}(\mathbf{s})$ may be sparse, there is no guarantee that $\mathbf{X}_1$ and $\mathbf{X}_2$ will be sparse as well — in fact, they are almost always dense. Obviously, for large-scale case where $q = 10^3$ or more, an optimization criteria based on the inequalities (14) and (15) would be prohibitive from the computational point of view. Instead, we propose a reduced-order optimization procedure as follows.

1. Determine the initial stiffness parameters $\tilde{\mathbf{s}}$ that satisfy the constraints (4).

2. For such constant $\tilde{\mathbf{s}}$, calculate a matrix $\mathbf{V} \equiv \mathbf{V}(\tilde{\mathbf{s}}) = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{pmatrix} \in \mathbf{R}^{q\times r}$, $r \ll q$, where $\mathbf{v}_i \mid i = 1, \ldots r$ are generalized eigenvectors of the matrix pair $(\mathbf{M}, \mathbf{S}(\tilde{\mathbf{s}}))$ that represent the structure critical vibration modes.

3. Apply a projection procedure to obtain reduced system matrices as follows

$$\begin{gathered}\mathbf{M}_r = \mathbf{V}^{\mathrm{T}}\mathbf{M}\mathbf{V},\ \mathbf{D}_{\mathbf{r}} = \mathbf{V}^{\mathrm{T}}\mathbf{D}\mathbf{V},\\ \mathbf{S}_r(\mathbf{s}) = \mathbf{V}^{\mathrm{T}}\mathbf{S}_0\mathbf{V} + \textstyle\sum_{i=1}^{l} s_i\mathbf{V}^{\mathrm{T}}\mathbf{S}_i\mathbf{V},\\ \mathbf{B}_{1r} = \mathbf{V}^{\mathrm{T}}\mathbf{B}_1,\ \mathbf{C}_{1r} = \mathbf{C}_1\mathbf{V},\ \mathbf{C}_{2r} = \mathbf{C}_2\mathbf{V}.\end{gathered} \tag{17}$$

4. Find a solution $\hat{\mathbf{s}}$ for the optimization problem (16) such that:

- stiffness parameters $\hat{\mathbf{s}}$ satisfy the constraints (4),
- there exist $\mathbf{X}_{1r} = \mathbf{X}_{1r}^{\mathrm{T}}, \mathbf{X}_{2r} \in \mathbf{R}^{q\times q}$ such that (14) and (15) hold true for the reduced system matrices (17).

Note that the matrix $\mathbf{V}$, which depends on the stiffness parameters $\mathbf{s}$, is calculated for some initial $\tilde{\mathbf{s}}$ and kept constant throughout the rest of the optimization procedure. This brings us to the crucial requirement for the proposed procedure: the matrix $\mathbf{V}$ does not depend on the parameters $\mathbf{s}$ significantly. More accurately, columns of $\mathbf{V}(\tilde{\mathbf{s}})$ span a subspace that is sufficient approximation of a column subspace of $\mathbf{V}(\hat{\mathbf{s}})$. This requirement, although not valid for the most general case, appears to be fulfilled for the vast majority of stiffness optimization problems we have encountered due to the following.

Let $\delta\mathbf{s} = \tilde{\mathbf{s}} - \hat{\mathbf{s}}$ denote the differences in initial and optimal stiffness parameters, respectively, and assume that $\delta\mathbf{s}$ are sufficiently small, i.e. the initial and optimal stiffness parameters are sufficiently close and/or the constraints (4) keep the parameters within some sufficiently small set. Consequently, columns of $\mathbf{V}(\hat{\mathbf{s}}) = \begin{pmatrix} \hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 & \cdots & \hat{\mathbf{v}}_r \end{pmatrix} \in \mathbf{R}^{q\times r}$ may be viewed as perturbed generalized eigenvectors

$$\hat{\mathbf{v}}_i = \sum_{j=1, j\neq i}^{q} \frac{\mathbf{v}_j^{\mathrm{T}}\left(\delta\mathbf{S}\right)\mathbf{v}_i}{\lambda_i - \lambda_j}\mathbf{v}_j, \tag{18}$$

where $\delta\mathbf{S} = \mathbf{S}(\hat{\mathbf{s}}) - \mathbf{S}(\tilde{\mathbf{s}})$ is stiffness matrix perturbation due to $\delta\mathbf{s}$, and $\lambda_i \mid i = 1, \ldots, q$ are generalized eigenvalues that correspond to the generalized eigenvectors $\mathbf{v}_i \mid i = 1, \ldots, q$ [6]. According to (18), columns of $\mathbf{V}(\tilde{\mathbf{s}})$ and $\mathbf{V}(\hat{\mathbf{s}})$ span the same subspace if the summation index q in the last term in (18) is replaced by r, or in other words, if the influence of the generalized eigenvectors $\mathbf{v}_j \mid j = r+1, \ldots, q$ on the perturbed generalized eigenvectors $\hat{\mathbf{v}}_i \mid i = 1, \ldots, r$ is neglected. Such influence is quantified by the constants

$$\epsilon_j = \frac{\mathbf{v}_j^{\mathrm{T}}\left(\delta\mathbf{S}\right)\mathbf{v}_i}{\lambda_i - \lambda_j}, \quad \text{for} \quad j = r+1, \ldots, q, \tag{19}$$

which are small if $\delta\mathbf{S}$ is small and $\lambda_i - \lambda_j$ is large, i.e. for the generalized eigenvalue/eigenvector pairs that are further from the eigenvector/eigenvalue pairs that represent structure critical vibration modes.

Therefore, the assumption that $\mathbf{V}(\mathbf{s})$ is (nearly) constant throughout the optimization procedure may be interpreted as follows: the changes in stiffness parameters do not cause significant contribution of the higher vibration forms to the critical vibration forms of the system. This may be verified a posteriori by evaluating the constants (19), or by checking the distance between $\mathbf{V}(\tilde{\mathbf{s}})$ and $\mathbf{V}(\hat{\mathbf{s}})$ by an appropriate measure, for example by calculating maximum singular value $\sigma_{\max}\left(\mathbf{V}(\tilde{\mathbf{s}}) - \mathbf{V}(\hat{\mathbf{s}})\right)$. If such measure is significant, i.e. if $\mathbf{V}(\tilde{\mathbf{s}})$ and $\mathbf{V}(\hat{\mathbf{s}})$ are substantially different, one may simply choose a larger number of vibration forms for the order reduction. Another alternative, which is the work in progress, is some sort of iterative procedure that consists of several reduction and optimization sequences.

IV. Numerical example

As an illustrative example of the applicability and efficiency of the proposed optimization framework, the following problem is studied. Consider a power plant comprising of a turbine and a generator connected by a shaft depicted in Fig. 1. The shaft is placed at two bearing blocks, reffered to bearing 1 and bearing 2, with radial stiffness parameters s_1 and s_2, respectively. The parameters for the power plant are: shaft lengths $a = 1938$ mm, $b = 7000$ mm, $c = 1310$ mm, shaft diameter $D = 900$ mm, turbine mass $m_T = 64000$ kg, turbine moments of inertia $I_{Tx} = 68000$ kg m^2, $I_{Tz} = 34000$ kg m^2, generator mass $m_G = 230000$ kg, generator moments of inertia

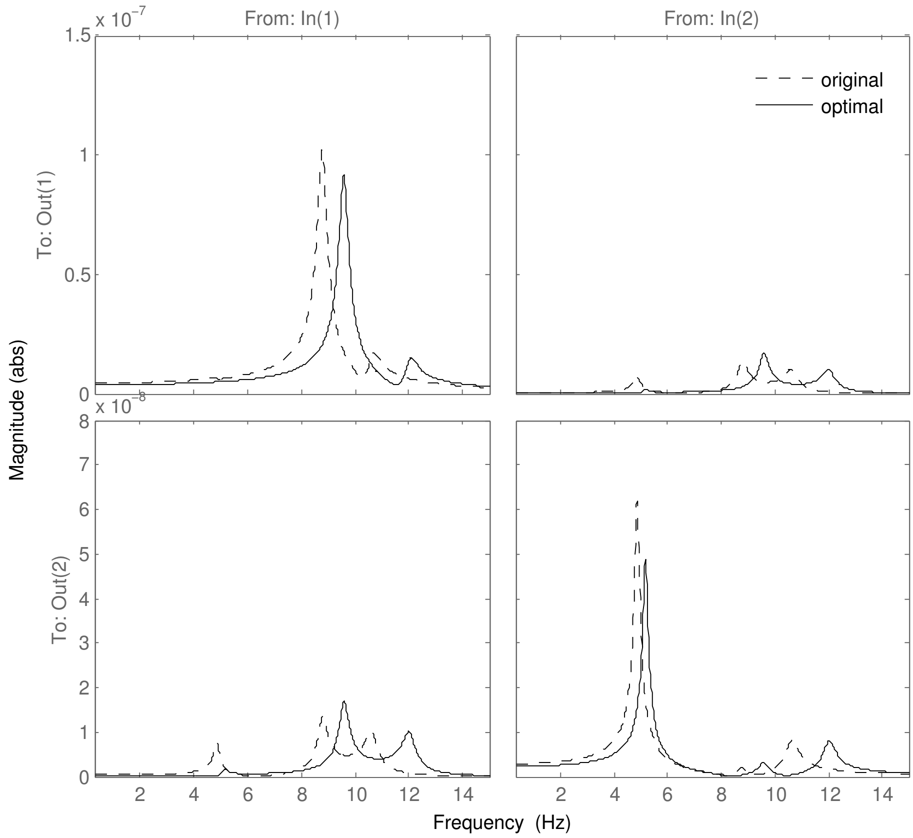

Fig. 2. Frequency response for original and optimized power plant vibration model

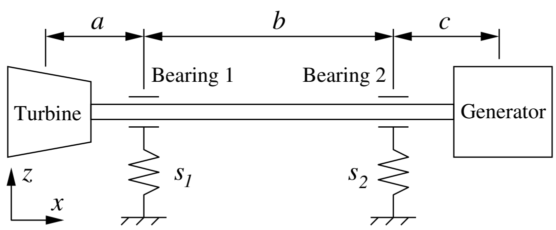

Fig. 1. Power plant

$I_{Gx} = 2 \cdot 10^6$ kg m^2, $I_{Gz} = 10^6$ kg m^2, bearing stiffnesses $s_1 = 0.6667 \cdot 10^9$ N m^{-1}, $s_2 = 0.6667 \cdot 10^9$ N m^{-1}.

The vibrations of the plant are excited by two harmonic forces $f_1(t) = f_{10} \sin(\omega t)$, reffered to as input 1, and $f_2(t) = f_{20} \sin(\omega t)$, reffered to as input 2, acting perpendicular to the shaft at the turbine (input 1) and the generator (input 2). For such vibration model, two outputs are defined as well: vibration displacements at the turbine and generator are reffered to as output 1 and output 2, respectively.

For the simulation and optimization purposes, we use a finite element model comprising of 10 shear deformable beam elements to model the shaft. The turbine and the generator are considered to be a discrete mass/inertia elements. Shaft material properties are as follows: modulus of elasticity $E = 210$ GPa, mass density $\rho = 7850$ kg m^{-3}, Poisson coefficent $\nu = 0.3$. Damping is proportional, with Rayleigh damping coefficients $\alpha = 0.8319$, $\beta = 4.2716 \cdot 10^{-4}$.

Mode	1	2	3	4
Frequency, Hz	4.85	8.78	10.62	18.46
Mod. damp., %	2.01	1.93	2.05	2.84

TABLE I. Results of the modal analysis

A frequency response of such model is shown as a dashed line in Fig. 2. A modal analysis of the structure is performed as well, and the results for the modal frequencies and modal damping ratios for the first four vibration modes are presented in Table I. Based on such results, we have identified the first three vibration modes as critical, i.e. most contributive to the system vibration response. Therefore, we construct the reduced order model using the projection matrices comprising of eigenvectors for the first three vibration modes, as described in section II.

In order to attenuate forced vibrations, we optimize turbine bearing stiffness parameters s_1 and s_2, taking into the account the following constraints:

$$\begin{aligned} 0.3333 \cdot 10^9 \text{ N m}^{-1} \leq s_1 \leq 1 \cdot 10^9 \text{ N m}^{-1}, \\ 0.3333 \cdot 10^9 \text{ N m}^{-1} \leq s_2 \leq 1 \cdot 10^9 \text{ N m}^{-1}. \end{aligned} \tag{20}$$

In other words, allowed bearing stiffness parameters range is between 50 % and 200 % of their initial values.

After the optimization, the following optimal bearing stiffness parameters are obtained: $s_1 = 0.93225 \cdot 10^9$ N m^{-1}, $s_2 = 0.93131 \cdot 10^9$ N m^{-1}. A frequency response for the optimized model is shown as a solid line in Fig. 2. As a result of the optimization, the peak frequency response has been reduced from $10.2 \cdot 10^{-8}$ to $9.13 \cdot 10^{-8}$, or in other words by 10.49 %, for turbine vibrations (output 1) due to turbine excitation (input 1). This peak frequency response corresponds to the second vibration mode, which has shifted from 8.78 Hz to 9.57 Hz due to stiffer optimal bearings. Frequency response for the generator vibrations (output 2) due to generator excitation (input 2) has been reduced from $6.11 \cdot 10^{-8}$ to $4.89 \cdot 10^{-8}$, or in other words by 19.97 %. This corresponds to the first vibration mode, which has shifted from 4.85 Hz to 5.17 Hz.

V. Conclusions

This paper presents the optimization framework based upon H_∞ optimality condition, which allows stiffness parameters optimization. The proposed approach assumes asymptotic stability of the structural system for all feasible stiffness parameters, which may be guaranteed a priori for majority of structural vibration problems.

It must be stressed that the main idea behind the proposed technique is local optimization, and consequently, its result is largely influenced by initial guess, stiffness parameters constraints, as well as the specific optimization problem. We do not consider this to be a serious drawback, since a good initial guess for stiffness parameters is often available. Finally, it may be argued that vast majority of the existing stiffness parameter optimization approaches cited in section I seek local minima as well.

To overcome its high computational cost and make it applicable for optimization of large-scale structures, we propose a novel reduced-order optimization algorithm. It comprises of modal projection of parametrized system matrices, and relies upon the assumption of small sensitivity of the subspace that represents critical vibration modes with respect to changes in stiffness parameters throughout the optimization. Note that any other reduction technique may be used instead, as long as it satisfies this assumption, and preserves the structure of the system matrices.

Appendix

I. LMI transformations

Lemma 1 (Elimination lemma) A linear matrix inequaltiy

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} & \mathbf{X}_{13} \\ \mathbf{X}_{21} & \mathbf{X}_{22} & \mathbf{X}_{23} \\ \mathbf{X}_{31} & \mathbf{X}_{32} & \mathbf{X}_{33} \end{pmatrix} < \mathbf{0} \tag{21}$$

holds true if and only if the following LMI's hold true

$$\begin{pmatrix} \mathbf{X}_{11} & \mathbf{X}_{13} \\ \mathbf{X}_{31} & \mathbf{X}_{33} \end{pmatrix} < \mathbf{0}, \tag{22}$$

$$\begin{pmatrix} \mathbf{X}_{22} & \mathbf{X}_{23} \\ \mathbf{X}_{32} & \mathbf{X}_{33} \end{pmatrix} < \mathbf{0}. \tag{23}$$

Proof: Since (21) holds true if and only if $\mathbf{x}^\mathrm{T}\mathbf{X}\mathbf{x} < 0$ for every $\mathbf{x}$, we partition $\mathbf{x} = \begin{pmatrix} \mathbf{x}_1^\mathrm{T} & \mathbf{x}_2^\mathrm{T} & \mathbf{x}_3^\mathrm{T} \end{pmatrix}^\mathrm{T}$ according to the block structure of $\mathbf{X}$. Equivalences (21) $\Leftrightarrow$ (22) and (21) $\Leftrightarrow$ (23) follow directly for $\mathbf{x}_1 = \mathbf{0}$ and $\mathbf{x}_2 = \mathbf{0}$, respectively. ■

References

[1] Bai Y. and Grigoriadis K. M. Damping parameter design optimization in structural systems using an explicit H_∞ norm bound. *Journal of Sound and Vibration*, 319:795–806, 2009.

[2] Zuo L. and Nayfeh S. A. Minimax optimization of multi-degree-of-freedom tuned-mass dampers. *Journal of Sound and Vibration*, 272:893–908, 2004.

[3] Zuo L. and Nayfeh S. A. Optimization of the individual stiffness and damping parameters in multiple-tuned-mass-damper systems. *Journal of Vibration and Acoustics*, 127:77–83, 2005.

[4] Scherer C. W. and Weiland S. Linear matrix inequalities in control. Lecture notes, Delft Center for Systems and Control, 2005.

[5] Kocvara M. and Stingl M. PENNON — A code for convex nonlinear and semidefinite programming. *Optimization Methods and Software*, 18(3):317–333, 2003.

[6] Cha P. D. and Qu W. Comparing the perturbed eigensolutions of a generalized and a standard eigenvalue problem. *Journal of Sound and Vibration*, 227(5):1122-1132, 1999.